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LE CALCUL PAR ITÉRATION DES POUTRES VIERENDEEL
DIE BERECHNUNG VON VIERENDEEL-TRÄGERN
DURCH ITERATION

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THE ANALYSIS OF VIERENDEEL TRUSSES BY SUCCESSIVE APPROXIMATIONS ¹⁾

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Introduction.

The term, Vierendeel truss or system, has been widely used to designate the rigid frame type of structure that is illustrated by the accompanying diagrams (Figs. 1—4). Since the diagonal members are omitted in these structures, their function must be taken over by the remaining members which are thereby subjected to flexural and shearing stresses in addition to the ordinary truss action. The two Vierendeel trusses that are shown in Figs. 1 and 2 are suitable for bridges, whereas Figs. 3 and 4 illustrate types of rigid frame systems that are used in viaduct and building construction. Before proceeding with an explanation of the analysis of these structures, however, it seems desirable to give at least a brief account of their history and development, and of various methods of solution, particularly the various methods of successive approximations that are gradually displacing the so-called exact methods of analysis.

History and Development: More than a century ago many statically indeterminate framed structures were used in the construction of bridges and buildings. A study of some of the bridges that were built before the development of the modern methods of rational analysis reveals a firm belief in the inherent strength of multiple system and rigid frame structures. The fact that several of these bridges are still in good condition apparently justifies the confidence of the builders. After the application of mathematical analysis to the design of structures, however, there developed a feeling of distrust and suspicion toward those structures that could not be rigorously analyzed and, as practical methods of analysis were limited in scope, most bridges were therefore built of a type that could be readily solved by the equations of static equilibrium. The type of structure most commonly used for bridges was the pin-connected or riveted articulated truss. The development of steel as a structural material also increased the use of this type of structure and so, at the present time, we find that a large number of the existing bridges belong to this class.

¹⁾ The major part of this paper is taken from a thesis submitted by the writer to the University of Michigan in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Engineering Mechanics.

However, with a gradual improvement in the methods of analyzing indeterminate structures and with the increased use of reinforced concrete, the rigid frame structure has again come into favor with many engineers, wherever conditions warrant their use. Since 1897, Professor Viereendeel and other Belgium engineers have successfully designed and constructed many bridges of the type shown in Figs. 1 and 2 for which they claim economy, strength, and dynamic rigidity²⁾. In an article on rigid frame structures, Engesser³⁾ points out that several bridges were built in Europe prior to 1800 that possessed many of the characteristics of the Viereendeel truss. However, several

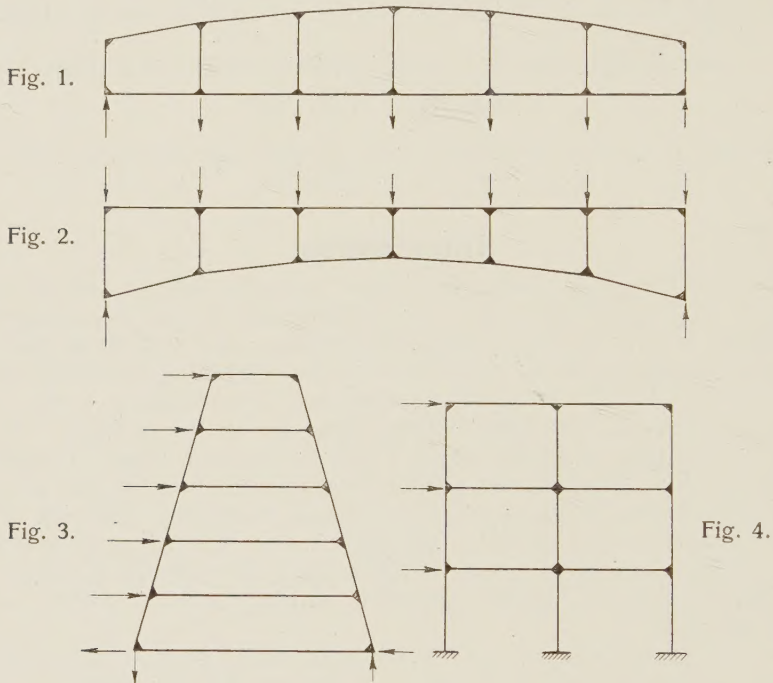


Fig. 1—4:
Types de systèmes Viereendeel. — Typen von Viereendeelsystemen.
Types of Viereendeel Systems.

of these structures, such as the Coalbrookdale bridge in England, were designed as voussoir arches and did not depend on the rigidity of the auxiliary bracing. Probably a better example of an early rigid frame bridge is the Wear river bridge⁴⁾ at Sunderland, England, which was designed by T. E. Harrison in 1879. This structure consisted of wrought iron girders 310 ft. long and 42 ft. deep at the center, with large openings in the web that formed panels 20 ft. in length. Also, in 1877, Thaddeus Hyatt⁵⁾ took out a patent in England on reinforced concrete bridges with large openings to reduce the weight of the structure. But, regardless of these earlier examples of similar

²⁾ The Viereendeel Truss by Dana Young, Eng. News-Record, Aug. 31, 1931.

³⁾ Über Rahmenträger und ihre Beziehungen zu den Fachwerkträgern. Zeitschrift für Architektur und Ingenieurwesen (1913).

⁴⁾ History of Bridge Engineering by Tyrrell, P. 183.

⁵⁾ Fachwerkträger aus Eisenbeton von S. Zipker, Eisen und Beton, 1906.

types of structures, it has been customary to refer to rigid frame trusses without diagonals as Vierendeel trusses.

Some Early Methods of Analysis: The rational analysis of the Vierendeel truss was first presented in a comprehensive form by Professor Vierendeel in various publications during the years 1897 to 1903⁶⁾. In the original presentation the equations were derived by considering the deformation in the members due to both bending and direct stress, but the complicated form of the equations made their application to actual structures very difficult. A simplified form of the equations was then recommended by Professor Vierendeel as being sufficiently correct for practical use. This simplification was obtained by considering only the deformation due to the bending moments and by assuming that all members have the same cross-sectional area and moment of inertia. These assumptions were used in writing an equation

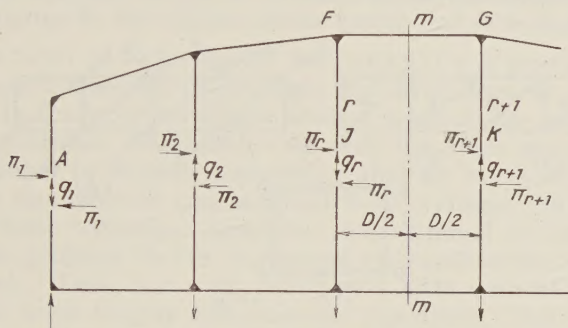


Fig. 5.

Méthode de calcul Vierendeel.
Untersuchungsmethode von Vierendeel.
Vierendeel's Method of Analysis.

for the displacement of the point of inflexion K of any vertical $r + 1$ (Fig. 5) with respect to A , the inflexion point of the first vertical, for both upper and lower portions of the structure. By equating the value of the displacement of any point K for the upper portion equal to that of the lower, one equation can be obtained for each vertical of the form:

$$\begin{aligned} \Pi_{r+1} &= C \Pi_r + C_1 \sum_1^r \Pi - C_2 M_r^{r+1} \text{ in which} \\ \Pi_{r+1} &= \text{Horizontal component in vertical } r + 1 \\ \Pi_r &= \text{Horizontal component in vertical } r. \end{aligned}$$

C , C_1 , C_2 are coefficients that are determined from the dimensions of the structure.

M_r^{r+1} = bending moment at section $m - m$ at the center of the panel.

Since one of the above equations can be written for each vertical in terms of the horizontal component Π_1 of the first vertical and since $\sum \Pi$ must be equal to zero, the value of the horizontal component for each vertical can be obtained.

Although this solution is not particularly difficult, it evidently did not satisfy all the requirements as it has not been very generally used outside of

⁶⁾ See "Theorie générale des poutres Vierendeel" in Mémoires de la Société des Ingénieurs Civils de France, 1900, also „Cours de Stabilité des constructions“, Vol. 4 by A. Vierendeel.

Belgium and France. Some of the objections that have been expressed are: Uncertainty of the effect of the various assumptions used in the derivations; the use of numbers with many digits in the equations to obtain an answer with relatively few; and also, its failure to meet the requirements for variable conditions in the arrangement of the structure.

Consequently, in both Europe and America, other methods of solution have been proposed and used. For instance, the Kinzua viaduct⁷⁾ built by the Erie Railroad in 1900, was designed by C. R. Grimm by the method of least work. By careful arrangement of the equations, the work was considerably reduced and numerical errors eliminated. However, the amount of work involved is many times greater than for the solution of the same structure that is given later.

In 1904 Professor L. F. Nicolai⁸⁾ in St. Petersburg, Russia, proposed a solution for Vierendeel trusses that is based on the assumption that the

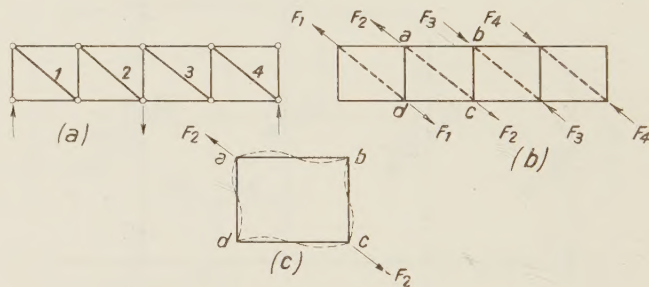


Fig. 6.

Méthode de calcul d'après Nicolai.

Untersuchungsmethode von Nicolai.

Nicolai's Method of Analysis.

rectangular panel composed of the upper and lower chords and the two verticals is the fundamental unit and that its relation to the remainder of the structure can be ignored. The analysis is then made by first assuming that diagonals are acting in the panels and that the joints are pin-connected as shown in Fig. 6 a. Then at each joint an equal external force is applied that will neutralize the stress in the diagonal (Fig. 6 b). The moments in each panel due to the external forces that are required to balance the stress in the diagonal are taken as the bending moments in the original structure (Fig. 6 c).

This method, used by Professor Nicolai, was greatly extended and made more practicable by Professor K. Čališev⁹⁾ in 1921 when he added to the primary moments in each panel, as obtained above, a correction in the form of a rapidly converging series which takes into consideration the restraining influence of the remaining portion of the structure. Additional corrections were also made for the change in length of the chords and for application of the load directly on the chord members. As several basic principles of this method will be utilized in the analytical solution that is described in the following chapters, a detailed description will not now be given.

⁷⁾ "The Kinzua Viaduct of the Erie R. R. Company", by C. R. Grimm. Trans. of A. S. C. E. 1901, Vol. 46.

⁸⁾ Journal of the Ministry of Ways of Communication, St. Petersburg, Russia.

⁹⁾ Dissertation presented by K. Čališev at the Polytechnical Institute at Zagreb, Yugoslavia for the Doctorate degree. Also, see Bauingenieur 1922, P. 244.

Several methods of solution for Vierendeel trusses have been published in Germany during the last thirty years. Such noted engineers as Engesser and Ostenfeld have contributed some very interesting and original solutions to this subject. In 1913 Engesser¹⁰⁾ presented an approximate method that was based on a primary solution of a hypothetical structure with rigid verticals, to which a correction for the change in length of the chords and for the bending of the verticals was added. The results that have been obtained by this method do not seem to be sufficiently accurate to compensate for the time that is required to make the calculations.

Professor Ostenfeld of Denmark¹¹⁾ has also presented an original method of solution which the engineer can sometimes use to advantage even though it is rather involved. This method depends upon the proper selection of the position of the unknown forces acting on the various members so that the fundamental equations may be simplified. In this respect the theory is somewhat analogous to the elastic center method of arch analysis.

Some Recent Methods of Successive Approximations: The so-called exact methods of analysis that have been so generally developed and used in academic work have always required the solution of several simultaneous equations. The number of such equations must, of course, be equal to the number of unknown quantities that are necessary to completely determine both the internal and external force systems. In the Vierendeel truss there are three redundant quantities in each panel so that $3n$ equations are required in addition to the equations of equilibrium, where n is the number of panels. Thus, for a six panel truss with vertical loads, it would be necessary to solve twenty simultaneous equations. This number can be materially reduced if the structure is symmetrical about some axis, but even so, the solution by these methods is very laborious. Consequently, the tendency in practice has been to reduce the number of equations by making various assumptions in regard to the position of the points of inflexion or the degree of restraint at the ends of the members. For instance, in the approximate methods for determining the wind stresses in tall buildings, sufficient assumptions are made to eliminate the use of simultaneous equations entirely.

At the present time, however, most engineers regard such indiscriminate use of assumptions with justifiable suspicion and are favoring the use of more accurate methods of solution that are based on the principle of successive approximations. These latter methods have one common characteristic in that certain strain conditions are assumed which give results that, at first, will not satisfy the conditions of equilibrium but, by successive trials, the computed values can be made to approach nearer and nearer to the true ones. The final values must, of course, satisfy both strain and equilibrium requirements. A brief description of several of these methods will be presented.

One of the first engineers to apply the method of successive approximation to the analysis of indeterminate structures was Mr. J. A. L. Waddell¹²⁾ who, in 1916, used it to determine the secondary stresses in bridge trusses. In his description of this method Mr. Waddell states that the solution is not

¹⁰⁾ „Die Berechnung der Rahmenträger mit besonderer Rücksicht auf die Anwendung“, by F. Engesser. Zeitschrift für Bauwesen, 1913.

¹¹⁾ „Beitrag zur Berechnung von Vierendeelträgern“, in Beton und Eisen, 1910. Eisenbau, 1912.

¹²⁾ See „Bridge Engineering“, Vol. 1, by J. A. L. Waddell.

original with him, but the writer does not know of any prior use. In this solution the moment at the end of any member MN is expressed in the usual form $M_{mn} = \frac{4EI}{l} (\tau_{mn} - \frac{1}{2} \tau_{nm})$ where τ_{mn} and τ_{nm} are the deflection angles of the tangents at the ends of the member. However, instead of obtaining the values of τ by the usual algebraic methods, they are determined by successive approximations. The change of slope of any member, ψ , is found from a displacement diagram of the truss, such as a Williot diagram. Therefore, only the joint rotation ϕ must be determined since $\tau = \phi - \psi$. For the first approximation, ϕ is taken as the average of the ψ 's of the two chord members meeting at the joint. When the end moments have been computed for these trial values of τ , it will ordinarily be found that the sum of the moments at the various joints will not equal zero so that some correction is necessary. This correction is then made for any joint by assuming that it is the only one that rotates, or that no other value of τ changes, which gives

$$\Delta \tau = \frac{\sum M \text{ at joint}}{4 \sum \frac{I}{l} \text{ at joint}}.$$

The effect of this correction on the various moments can be carried over to other joints before they are corrected, which procedure makes the convergence very rapid, particularly when the joints with the greatest unbalance are considered first. The final results can be carried to almost any degree of accuracy, but in general two cycles of corrections will be sufficient. Many variations of this method can and have been used but they ordinarily differ only in the arrangement of the numerical calculations.

The above method for the determination of secondary stresses in bridge trusses was also used by Professor K. Čališev¹³⁾ who modified the equations to take into account the decrease in the effective length of the member due to the greatly increased area at the joints. In 1923, Professor Čališev applied a similar method¹⁴⁾ of successive approximations to the solution of rigid frame structures. In this method the solution was divided into two parts; first, all translation of the joints was neglected so that only rotation was involved, and second, the effect of translation was then considered so as to satisfy the necessary strain and equilibrium conditions. When only rotation of the joints are considered, the end moments can be expressed by equations of the form $M_{ab} = 2K(2\theta_a + \theta_b) + M_{fab}$ in which θ 's represent the joint rotation, $K = EI/l$ and M_{fab} is the end moment produced by the transverse loads for a fully restrained condition, or what is commonly called the fixed-end moment. As in the above method for secondary stresses, the proper value for the various θ 's can be determined by successive approximations. If all joints are held motionless except one, then the rotation of that joint will be $\theta = \frac{\sum M_f}{4 \sum K}$. The effect of this correction on the end moments at other joints can be considered before those joints are corrected so that, by careful ar-

¹³⁾ "Secondary Stresses in Trusses", by K. Čališev, Eng. Journal of the Soc. of Eng. and Arch. of Yugoslavia (1922).

¹⁴⁾ Analysis of highly statically indeterminate systems by the method of successive approximations by K. Čališev, Eng. Journal of the Soc. of Eng. and Arch. of Yugoslavia (1923).

rangement, the convergence can be made very rapid. If the members are hinged at one end then $3/4 K$ should be used instead of the full value.

The second part of the problem which involves the correction for the translation of the joints can be made in a manner similar to the solution

Fig. 7a.

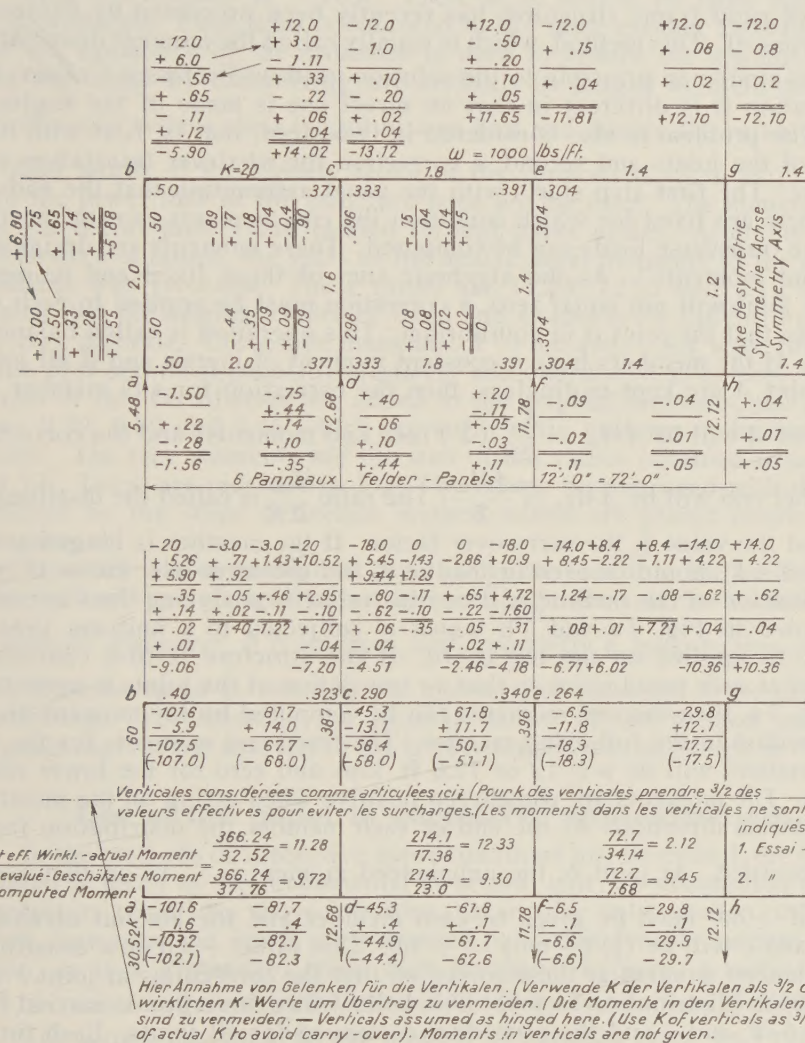


Fig. 7b.

Méthode de calcul d'après Cross.
Untersuchungsmethode von Cross.
Cross's Method of Analysis.

described above for secondary stresses except that now both ψ and Θ are unknown. However, for frames of one story, any value of ψ can be assumed and the moments thus computed will be in the same proportion to the true moments as the computed shear is to the true shear. If the structure has more

than one story, this proportionality does not hold and consequently the solution becomes much more involved. As Professor Čališev has presented this method in detail in another paper of this publication it will not be necessary to describe the numerical operations here.

Another application of the method of successive approximations to the analysis of rigid frame structures has recently been presented by Professor Hardy Cross¹⁵). This method, which is usually called the moment distribution method, is similar in principle to the solution used by Professor Čališev, but in application it is different in that no direct use is made of the angles θ and ψ . The problem is also considered in two steps, that is, first with only rotation of the joints and second, a correction for whatever translation may take place. The first step starts with the usual assumption that the ends of all members are fixed for which condition the end moments in each member due to the transverse loads can be computed. These moments are designated "fixed end moments". As the algebraic sum of these fixed end moments around a joint will not equal zero, a correction must be applied to each end moment so that the joint is in equilibrium. This operation is called balancing the joint. If the members have a constant moment of inertia and if all joints except joint A are kept motionless, then the correction for any member AB

at the joint A will be $\Delta M_{ab} = \frac{K_{ab}}{\sum K} (\sum \text{Fixed end moments})$ and the correction

at the other end will be $\Delta M_{ba} = \frac{\Delta M_{ab}}{2}$. The ratio $\frac{K_{ab}}{\sum K}$ is called the distribution

factor and the value $\frac{1}{2}$, the carry-over factor. If the member is hinged at the end then $\frac{3}{4} K$ should be used instead of K and the carry-over factor is zero. The application of this method will be illustrated by analyzing the Vierendeel truss shown in Fig. 7a and 7b which is subjected to a uniform load of 1000 lbs./ft. applied on the top chord. If the structure is first considered supported at each panel point so that no translation of the joints is permitted, as in Fig. 7a, then the end moments can be computed by the moment distribution method in the following manner. The fixed end moments for the top chord members will be $w l^2/12$ or 12.0 ft. kips and zero for the lower chord members. The moments are considered positive when acting on the members in a clockwise direction. At the end of each member the distribution factor

$\frac{K}{\sum K}$ is recorded. At joint b , the unbalanced moment is -12 so that a correction of $+6.0$ must be given to each member and the amount carried to joints a and c will be $(\frac{1}{2})(+6.0) = +3.0$. This value $+3.0$ now constitutes the unbalanced moment at these joints, so that the corrections at joint c will be $(+3.0) \times (\text{distribution factor})$. One half of this correction is carried over to joints b , d , and e and the procedure continued for all joints. Each time a joint is balanced a horizontal line is drawn under the figures. When the correction is small, the column of figures can be added and the final moment obtained. The value of the reactions that are necessary to prevent translation of the joints can be computed from the shear in the members.

The second step in the solution will be the calculation of the moments due to forces equal and opposite to the joint reactions, as shown in Fig. 7b.

¹⁵) "Analysis of Continuous Frames by Distributing Fixed end Moments", by Hardy Cross. Trans. of Am. Soc. Civ. Eng., Vol. 96, 1932.

The procedure used will be that recommended by Mr. L. E. Grinter¹⁶⁾. In this solution the fixed end moments in the chord members due to a vertical displacement A will be equal to $M_f = 6EK \frac{A}{l}$, so we must first assume some value for $\frac{A}{l}$. This value is first taken equal to $\frac{10}{6E}$ for each member, so that the fixed end moment is $-10K$. There is, of course, no fixed end moment in the verticals. Each joint is balanced in the same manner as before except that the symmetry of the structure is used to reduce the amount of work. This was done by considering the verticals as hinged at their mid points, which would reduce their length by one half and double the value of $K = \frac{EI}{l}$.

In other words, if the rigidity factor of the verticals is taken as $(3/4) (2K)$, then the lower half of the structure need not be used. The numerical work was started at joint c and carried through three cycles. After the end moments were obtained, the ratio of the actual panel moments to the computed moments was recorded. These ratios are 11.28, 12.33 and 2.12 which indicates that the original choice of fixed end moments for the third panel was not very accurate. Corrective moments of -3.0 and -8.4 were then placed in the first and third panels respectively and distributed as before. When the results of this distribution are added to the original moments, the ratios become 9.72, 9.30 and 9.45 which are sufficiently uniform to insure accurate results. The true moments are the sum of the values obtained from Fig. 7a and 7b. These are recorded in Fig. 7b and can be compared with the results obtained by the slope deflection method which are placed below them in parentheses.

From the above example it can readily be seen that the method can be used for any type of rigid frame structure regardless of the variation in the $\frac{I}{l}$ factors. However, it is indirect, and frequently a considerable amount of time must be spent in securing the correct end moments to be used in the second step. Nevertheless, the general nature of the method makes it a valuable tool for the analysis of the type of structures considered in this paper.

Use of the panel in successive approximations.

In the methods of successive approximations just described, the individual member is taken as the fundamental structural unit; an assumption that gives satisfactory results for those solutions that involve only rotation of the joints. When, however, the joints are subjected to a considerable amount of translation, as in a Vierendeel truss, then the panel becomes a more important unit, as was recognized in the methods used by Nicolai and Čališev.

The writer has changed the method of solution given by Prof. Čališev so as to make it more direct and also to bring it more in accordance with the ordinary use of methods of successive approximations, such as the moment distribution method. The primary difference consists in expressing the fundamental equations directly in terms of the external force system as shown in Figures 8a and 8b rather than in terms of the internal force system of an articulated truss as shown in Figures 6b and 6c. Also, the writer has made

¹⁶⁾ "Wind Stress Analysis Simplified", by L. E. Grinter. Proc. A.S.C.E., Jan. 1933.

no attempt to express the equations in terms of modified rigidity factors so as to obtain the final result with one cycle of convergence, but prefers to use the actual rigidity of the members with as many cycles of convergence as are necessary. The use of this method in analyzing some common types of Vierendeel systems will now be explained.

Notation.

K = rigidity factor of top and bottom chords = $\frac{\text{Moment of Inertia}}{\text{Length}}$.

K_1 and K_2 = rigidity factors of verticals in panel.

r = ratio $\frac{K}{K_1}$

s = ratio $\frac{K}{K_2}$

h_1 = length of short vertical of any panel.

h_2 = length of long vertical of any panel.

a = ratio $\frac{h_2 - h_1}{h_1}$.

M_{ad} and M_{da} = moments at ends of member ad .

M = bending moment in truss at beginning of panel.

V = total shear in panel.

L = panel length.

$D = 6 + r + s + \alpha (2a + \alpha s + 2s + 6)$.

ΔM = correction to primary moment.

m' and m'' = moments in adjacent panels.

H_{bc} , V_{bc} , M_{bc} = normal and shearing force and moment at (b) end of member bc .

A = cross-sectional area of top and bottom chord.

θ and β = slope angles of chords.

Δ = linear displacement of joint.

E = modulus of elasticity.

Trusses with chords of equal rigidity:

Let us first consider the moments in any panel of the truss shown in Fig. 8a, such as $abcd$. This panel will first be considered as hinged to the remainder of the structure and will therefore act as an independent frame. The forces acting on such a panel are shown in Fig. 8b, in which M is the bending moment in the truss at the section ab , V the external shear in the panel and $M + VL$ is the moment at section dc . The internal moments at the ends of the members for this force system will be designated the primary moments, and for trusses with chords of equal rigidity K , and verticals of rigidity K_1 and K_2 , these primary moments can be computed from the following equations:

$$\begin{aligned} M_{ad} = M_{bc} &= \frac{\alpha M - VL}{2} \left[\frac{3 + s + \alpha(2 + s)}{D} \right] \\ M_{da} = M_{cb} &= \frac{\alpha M - VL}{2} \left[\frac{3 + r + \alpha}{D} \right] \end{aligned} \quad \text{Eq. 1}$$

in which $D = 6 + r + s + \alpha (2a + \alpha s + 2s + 6)$

$$r = \frac{K}{K_1}, \quad s = \frac{K}{K_2}, \quad \alpha = \frac{h_2 - h_1}{h_1}, \quad L = \text{Panel Length.}$$

In the derivation of these equations only the deformation due to bending moments was considered. Both the external moment M and the internal moments are taken as positive when acting in a clockwise direction. By means of Eq. 1 the primary moments can be determined for each panel and recorded in a table or, preferably, on a sketch of the structure.

The restraining action of the adjacent panels will be considered next by analyzing the effect of moments applied to the panel $abcd$ as shown in Figs. 9a and 9b. These diagrams show the moments m' and m'' which exist in the panels below and above the panel $abcd$ and which represents the restraining action at the joints. The end moments in the panel $abcd$ produced

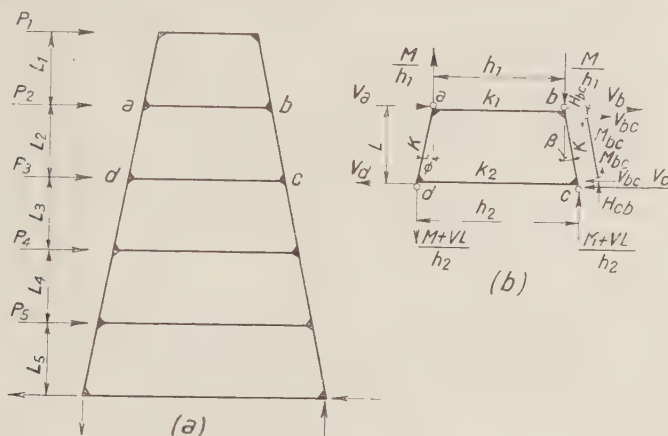


Fig. 8a und b.

Contraintes primaires dans un panneau.
Primäre Spannungen in einem Feld.
Primary action in a Panel.

by m' and m'' will be the required corrections and will be called the secondary moments. These secondary moments can be computed by the following formulae, Eqs. 2 and 3.

$$1 M_{ad} = \Delta M_{bc} = -\frac{s(1+\alpha)}{D} m' \quad \text{Eq. 2}$$

$$\Delta M_{da} = \Delta M_{cb} = \frac{s(1+\alpha)^2}{D} m'$$

$$\Delta M_{ad} = \Delta M_{bc} = \frac{r}{D} m''$$

$$\Delta M_{da} = \Delta M_{cb} = -\frac{r(1+\alpha)}{D} m'' \quad \text{Eq. 3}$$

In these equations m' and m'' are given a positive sign when acting clockwise on the panel. The corrections due to m' and m'' can be recorded on a sketch of the structure for each panel. Eqs. 1, 2 and 3 can be applied to trusses of the type shown in Figs. 1 and 2 as well as those shown in Fig. 8.

Numerical example: The application of the above equations will be illustrated by a numerical example. The bent shown in Fig. 10 was analyzed by Mr. C. R. Grimm in his description of the Kinzua Viaduct so that a com-

parison can be made with the results obtained by the method of least work. The first step in the solution was to tabulate the values of M , V , r , s , u and D for each panel on a sketch of the structure (Fig. 10). The primary moments were then computed by means of Eq. 1 and recorded in row 1. By using these primary moments in Eqs. 2 and 3, the first set of corrections were obtained. These corrections were made by going from top to bottom and then back as shown by the arrows. This first set of corrections can then be used to obtain additional corrections although in general these will be very small. The sum of the primary and secondary moments gives the final result as recorded in row 3. It will be noted that in this first solution the rigidity of the bottom strut is taken equal to infinity which is equivalent to the fixed end condition that Mr. Grimm assumed. The values of the moments that were obtained by Mr. Grimm are given in row 4. The moments recorded in row 5 were computed by using the actual rigidity of the bottom strut and by assuming the base of the columns as hinged. Because of the unknown restraint at the column base, the actual moments will be between these two results.

Effect of Change in Length of Chords: In deriving equation 1 for the primary moments, the effect of the change in length of the various members was neglected. This change in length occurs chiefly in the chord members as they are subjected to the greatest axial forces. A method for determining the primary moments in any panel, such as Fig. 8b, will now be given in which the deformation due to both bending moments and change in length of the chords is considered. The internal forces acting in the frame must be of such magnitude as to make the total strain energy a minimum. Thus, if an expression for the strain energy U in the frame is written in terms of H_{bc} , V_{bc} , M_{bc} , the forces at the end of the chord bc , an equation will be obtained in which $U = \Phi(H_{bc}, V_{bc}, M_{bc})$. For a condition of minimum strain energy, the requirement is that

$$\frac{\partial U}{\partial M_{bc}} = 0, \quad \frac{\partial U}{\partial H_{bc}} = 0, \quad \frac{\partial U}{\partial V_{bc}} = 0$$

which gives the following equations:

$$(a) \quad BM_{bc} - C \frac{H_{bc} h_1}{2} - V_{bc} L (1 + s) = -\frac{BM}{2} - \frac{VL}{2} (1 + s)$$

$$(b) \quad C \frac{M_{bc}}{2} - E \frac{H_{bc} h_1}{3} - F \frac{V_{bc} L}{6} = -\frac{M}{6} [6 + 2r + s(2 - \alpha)] -$$

$$\frac{VL}{6} [3 + s(2 - \alpha)] \quad \text{Eq. 4}$$

$$(c) \quad M_{bc}(1 + s) - F \frac{H_{bc} h_1}{6} - G \frac{V_{bc} L}{3} = -\frac{M}{2} (1 + s) - \frac{VL}{6} (2 + 3s)$$

$$\text{in which} \quad B = 2 + r + s \quad C = B - \alpha(1 + s)$$

$$E = (3 + r + s) + \alpha^2(1 + s) - \alpha s + \frac{3lK(\cos \Phi + \cos \beta)}{h_1^2 A}$$

$$F = 3(1 + s) - \alpha(2 + 3s) + \frac{6K(\sin \Phi + \sin \beta)}{h_1 A}$$

$$G = 2 + 3s + \frac{3K}{lA} \left(\frac{\sin^2 \Phi}{\cos \Phi} + \frac{\sin^2 \beta}{\cos \beta} \right).$$

The terms r , s and α are the same as before, while ϕ and β are the slope angles of the chords whose cross-sectional areas are A .

If the terms in the above equations that contain the cross-sectional area A are omitted, then Eq. 4 will reduce to Eq. 1, which neglects the effect of the direct stress.

The difference between the results of Eq. 1 and Eq. 4 can be shown by a numerical example. Let $r = 1$ $s = 1.5$.

$$\begin{aligned} \alpha &= .6 & h_1 &= 10 \text{ ft.} & L &= 12 \text{ ft.} & \cos \phi &= \frac{2}{\sqrt{5}} & \sin \phi &= \frac{1}{\sqrt{5}} \\ \beta &= 0 & A &= 10 \text{ sq. in.} & & & K &= 3.0 \text{ in.}^3. \end{aligned}$$

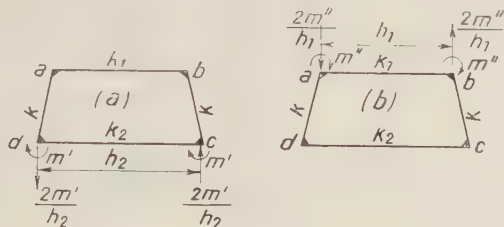


Fig. 9.

Contraintes dues aux panneaux adjacents.
Spannungen von Seite der benachbarten Felder.
Restraining Action of adjacent Panels.

For the above data Eq. 1 gives

$$M_{bc} = .1306 M - 2.612 V$$

$$M_{cb} = .0910 M - 1.820 V$$

while Eq. 4, which includes the effect of the direct stress, gives:

$$M_{bc} = .1332 M - 2.589 V$$

$$M_{cb} = .0886 M - 1.841 V$$

which shows that the effect of the direct stress is to decrease the moment at the h_1 end an amount $.0026 M + .023 V$ and to increase the moment at the h_2 end an amount $-.0024 M - .021 V$. It would therefore appear that in inclined chord trusses of reasonable proportions, the moments due to the change in length of the members can be neglected.

When the chord members are parallel, $\alpha = 0$, and equation 4 then reduces to the expression:

$$\begin{aligned} M_{bc} &= \frac{M \left(\frac{12 L K}{h^2 A} \right) - \frac{V L}{2} (3 + s)}{6 + r + s + \frac{24 L K}{h^2 A}} \\ M_{cb} &= \frac{-M \left(\frac{12 L K}{h^2 A} \right) - \frac{V L}{2} (3 + r + \frac{24 L K}{h^2 A})}{6 + r + s + \frac{24 L K}{h^2 A}} \end{aligned} \quad \text{Eq. 5}$$

If the same numerical data that was used above, except that $\alpha = 0$ instead of $.6$, is substituted in Eq. 5, the value of the moments become:

$$M_{bc} = .0042 M - 3.149 V$$
$$M_{cb} = -.0042 M - 2.851 V$$

where as from Eq. 1, we obtain the values:

$$M_{bc} = -.3.176 V \quad M_{cb} = -.2.824 V$$

which gives the effect of the direct stress as:

$$\Delta M_{bc} = .0042 M + .027 V \text{ and } \Delta M_{cb} = -.0042 M - .027 V$$

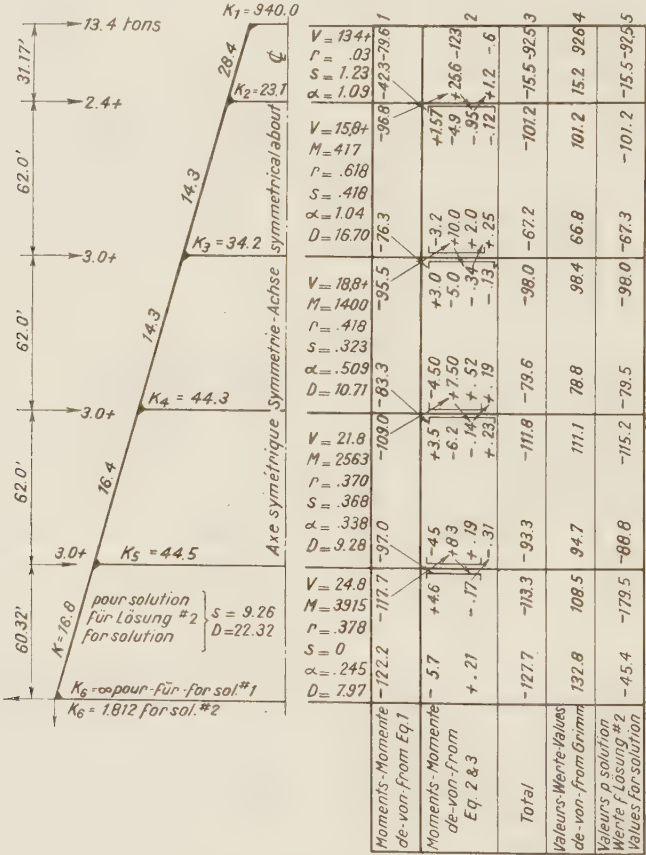


Fig. 10.

(Calcul du pont Kinzua d'après la méthode des panneaux.
Berechnung der Kinzua-Brücke nach der Feldermethode.
Analysis of the Kinzua Viaduct by the Panel Method.

A comparison of these results with the values previously obtained for $a = .6$ shows that the effect of the change in length of the chords is more important in parallel chord trusses than with inclined chord trusses. From equation 5, it can be seen that the change in length of the chords will have a more important effect for small values of h , or for shallow trusses with relatively long panels. This condition is seldom encountered in practice, but if it is, the primary moments can be easily obtained by using Eq. 5 instead of Eq. 1.

Effect of Truss Proportions: The term truss proportions refers here to the height of the verticals and length of panels but does not include the relative rigidity of the various members. By means of equation 1, we can see that the primary moments in any panel will be zero whenever $aM - VL = 0$. Since $a = \frac{h_2 - h_1}{h_1}$, this expression can be written, $\frac{h_2 - h_1}{L} = \tan \theta = \frac{V}{H}$ where $H = \frac{M}{h_1}$. This condition will evidently be satisfied when

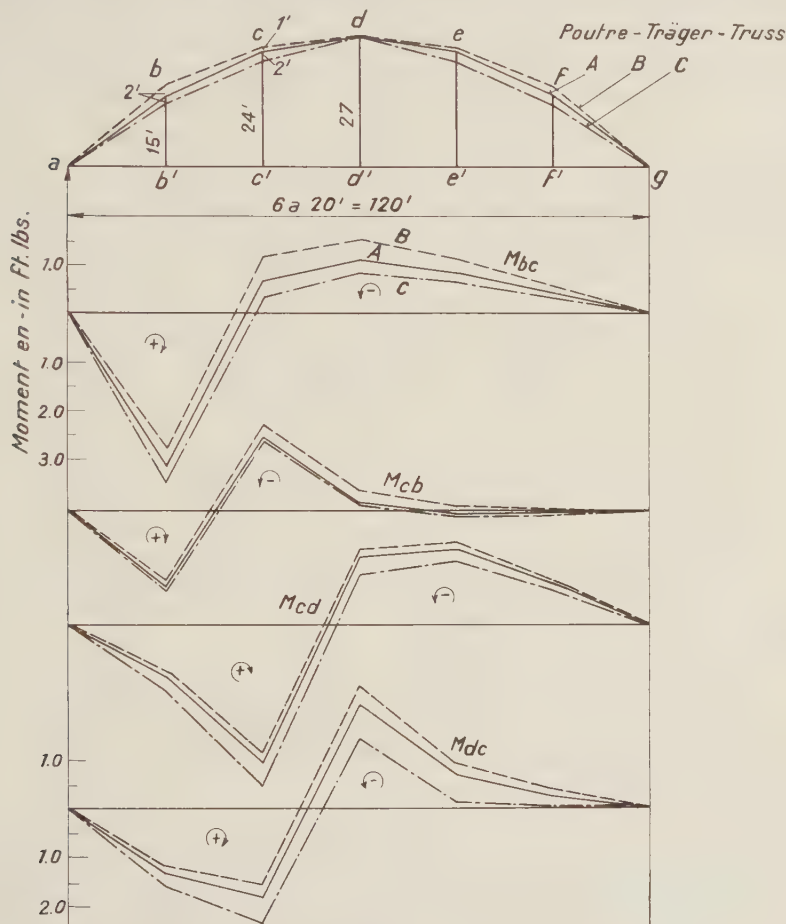


Fig. 11.

Lignes d'influence pour moments fléchissants dans des poutres de forme différente.
Einflußlinien für Momente in Trägern verschiedener Form.
Influence diagrams for Moments in various Trusses.

the axis of either chord coincides with the equilibrium polygon for any load system since the slope of the equilibrium polygon is equal to $\frac{V}{H}$. If the applied load is uniformly distributed over the span as is usually the case for the dead load, the equilibrium polygon is a parabola and, therefore, a parabolic curve for the top or bottom chords will be the most economical. The variation in

the moments of the chords when the axis of the top chord is moved above or below a parabola can be readily seen from the influence diagrams shown in Fig. 11. For truss *A*, in which the joints of the top chord lie on a parabola, the positive and negative areas of the influence lines are equal, while for trusses *B* and *C*, they are unequal.

A comparison of the total combined dead and live load moments for the three trusses of Fig. 11 has been given in Table 1. These moments were computed from the influence lines of Fig. 11 for a dead load of 1500 lbs. per linear foot of truss and a live load of the American *H*—20 standard highway loading plus 30 % for impact. It can be seen that the maximum moments for trusses *B* and *C* may be 50 to 60 % higher than the maximum moments for truss *A*. With a reduced dead load and a larger concentrated

Table I

Summary of moments for dead load of 1500 lbs. per foot of truss and American standard *H*—20 highway loading.

Moment	Truss	Dead Load	Live Load + 30% Impact		Maximum Combined
			+	—	
M_{bc}	A	0	140.5	73.0	+ 140.5
	B	— 48.7	125.0	97.8	— 146.5
	C	+ 37.5	154.0	53.2	+ 191.5
M_{cb}	A	0	70.8	69.4	+ 70.8
	B	— 25.3	66.1	82.3	— 107.6
	C	+ 5.8	71.5	64.5	+ 77.3
M_{cd}	A	0	122.5	94.1	+ 122.5
	B	— 17.6	113.8	101.6	— 119.2
	C	+ 45.8	147.2	75.8	+ 193.0
M_{dc}	A	0	87.5	93.6	— 93.6
	B	— 35.0	72.5	115.3	— 150.3
	C	+ 68.3	112.1	53.6	+ 180.4

live load, truss *B* might show a more favorable comparison, but truss *C* is unsatisfactory for any practical use.

If all panel lengths in a truss are the same, and if other factors are kept constant, then Eq. 1 shows that the moments are directly proportional to the panel length. For example, if the panel lengths in Fig. 11 are changed from 20 ft. to 30 ft., the moments given should be multiplied by 3/2. However, if the panel lengths are not equal then of course this proportion no longer holds.

An approximate solution for wind stresses in tall building frames.

The subject of wind stresses in building frames has received much attention in current technical literature and has brought forth much controversial discussion with respect to the use of those approximate solutions that depend on certain arbitrary assumptions in regard to the position of the points of contraflexure in beams and columns. Many engineers contend that a high degree of mathematical accuracy for such analysis is hardly justified in view of the inaccuracies that are involved in determining the maximum wind pressure on any building and in the allocation of the load to the various parts of

the building. They also point out that some analysis of the structure must be made before the design can be completed, and that certain approximate methods give sufficiently accurate results for this purpose. One of the approximate methods that is commonly used involves the following steps¹⁷⁾:

a) Distribute the total shear in each story to the columns so that the exterior columns have one-half the shear of the interior columns, or in some proportion determined by empirical rules.

b) Select the position of the points of contraflexure at the mid-points of the columns in all stories except the upper and lower ones where they are

Fig. 12a.

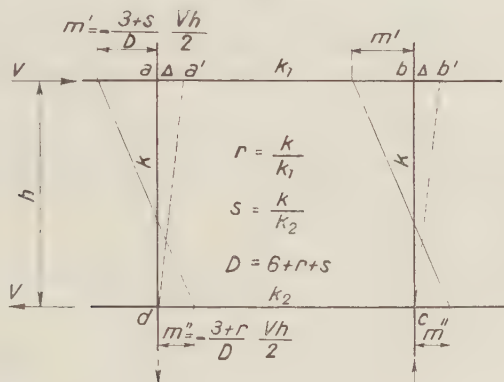


Fig. 12b.

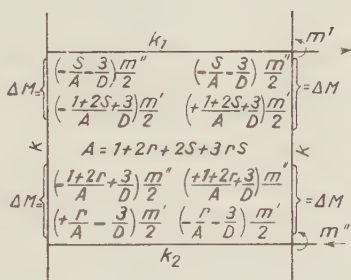
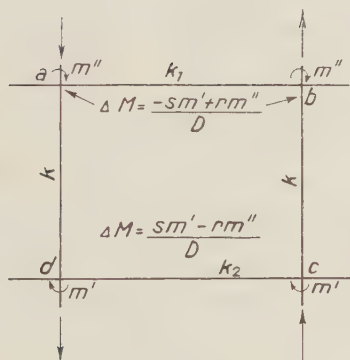


Fig. 12c.

Fig. 12a—c.

Formules pour le calcul des contraintes dues au vent dans la construction des bâtiments de grande hauteur d'après la méthode des panneaux.

Formeln zur Berechnung der Windspannungen in hohen Gebäuden nach der Feldmethode.

Formulae for the Calculation of Wind Stresses in Tall Buildings by the Panel Method.

taken at .65 of the height from the top for former and .6 of the height from the bottom for the latter.

c) Select the position of the points of contraflexure in the exterior girders at .55 of their length from the outer ends and at the mid-point of other girders unless the conditions of symmetry or equilibrium require otherwise.

The above procedure will usually give satisfactory results for buildings of regular proportions unless there is a sudden variation in the rigidity of

¹⁷⁾ See "Structural Theory", by Sutherland and Bowman.

the girders at consecutive floors. For the portion of a frame where such a variation may occur, the panel method that has just been applied to the Vierendeel truss can be used, since a building frame with lateral loads may be considered as a series of vertical trusses. The application of this method to any portion of a building frame in which the columns have approximately the same rigidity can be made in the following manner:

1. Each story of the bent, except the lower, is assumed to consist of a series of separate panels, such as $abcd$, Fig. 12 a, each one of which must resist a part of the total shear in the story. Each interior column will therefore form a chord of two different panels and the sum of the moments that are computed for the separate panels will be taken as the primary moment in the column. This moment must, of course, be corrected for the action of the adjacent panels in the same manner that was used in the preceding problems.

To determine the amount of the shear V that is taken by each panel, the usual assumption that the top of each panel in the story moves horizontally the same amount Δ (Fig. 12 a) will be made. This movement Δ can be expressed for each panel by the equation,

$$\Delta = \frac{h^2}{12E} \left[\frac{3 + 2r + 2s + rs}{K(6 + r + s)} \right] V \text{ or } \frac{12E\Delta}{h^2} = CV$$

where

$$C = \frac{3 + 2r + 2s + rs}{K(6 + r + s)}$$

and K = the average rigidity factor of the two columns of the panel. In other words, the value of CV must be constant for each panel in the story, which gives the relation

$$C_1 V_1 = C_2 V_2 = \dots C_n V_n$$

Also, the sum of the shears V for the panels in the story must be equal to the total shear in the story, or

$$V_1 + V_2 + \dots V_n = Q = \text{total shear in story.}$$

From these two conditions the shear in each panel can be computed.

2. Determine the moments in the columns by means of the following steps:

a) Calculate the primary moments m' and m'' (Fig. 12 a) for the columns in each panel by means of the equations

$$m' = -\left(\frac{3 + s}{D}\right) \frac{Vh}{2} \quad \text{and} \quad m'' = -\left(\frac{3 + r}{D}\right) \frac{Vh}{2}.$$

b) Calculate the vertical correction ΔM (Fig. 12 b) due to the action of the panels above and below.

c) Calculate the horizontal correction ΔM (Fig. 12 c) in the outer panel due to the action of the first interior panel. The horizontal correction in the other panels will ordinarily be small. If the columns are much more rigid than the beams, the horizontal correction can usually be ignored.

3. Determine the moments in the beams by assuming the position of the points of contraflexure as given above, or by distributing the sum of the column moments at any joint to the connecting girders in proportion to their rigidity.

The use of the above methods for calculating wind stresses in tall building frames will be illustrated by analyzing a portion of a bent of the

Buhl Building in Detroit (Fig. 13). From the values of the rigidity factors $\frac{I}{L}$ that are given, it can be seen that there is a sudden variation in the rigidity of the beams at the 23rd floor. The stories above and below this floor will

$Q = 20.88^k$								
11'-3"	3.70	-17.6 (16.0)	1.71	6.56	-35.2 (37.9)	1.76	Symmetrical about Φ	
8.33k 25 Floor		-27.4 (29.0)			-54.8 (58.3)			
11'-3"	3.70	-27.4 (24.4)	1.71	49.7 (50.4)	-54.8 (53.0)	51.2 (50.6)	1.76	Symmetrical about Φ
8.33k 24 Fl.		-26.8v +10.2v -4.9v -2.8h -24.3 (27.0)	$r = 4.24$ $s = 1.10$ $C = .222$ $V = 13.22$ $A = 25.7$	-26.8v +10.2v -4.9v -1.7h Total 115.5 (112.3)	-25.8 +9.7v -6.8v -46.1 (47.6)	$r = 4.51$ $s = 2.27$ $C = .264$ $V = 11.10$		
11'-3"	6.56	-47.5 -10.2v +4.9v -3.2h -56.0 (55.7)	6.62	-47.5 -10.2v +4.9v +7.6h Total 79.4	-36.7 -9.7v +6.8v -34.8 (60.4)			Symmetrie - Achse
8.33k 23 Fl.		-50.3 +5.7v -11.7v -3.9h -60.2 (61.6)	$r = 1.10$ $s = 2.13$ $C = .176$ $V = 16.1$ $A = 14.46$	-50.3 +5.7v -11.7v +5.9h Total 79.4	-38.3 +3.0v -91.7 (90.9)	61.0 (58.1)		
11'-3"	6.56	-40.1 -5.7v +11.7v -3.3h -37.4 (38.0)	3.40	-40.1 -5.7v +11.7v +1.4h Total 83.5	-39.3 +3.0v -68.0 (67.5)	$r = 2.28$ $s = 2.28$ $C = .208$ $V = 13.61$		Axe symétrique
8.33k 22 Fl.		-50.8 (56.2)		-101.6 (89.1)	86.1 (88.2)			
11'-3"	7.93	-50.8 (43.6)	3.40	10.87	-101.6 (93.6)	3.49		
8.33k 21 Fl.								
11'-3"	7.93			10.87				

Fig. 13.

Calcul des contraintes dues au vent d'après la méthode des panneaux.

Berechnung von Windbeanspruchungen durch die Feldmethode.

Calculation of Wind Stresses by the Panel Method.

therefore be analyzed by the panel method while the first approximate method will be used for the other stories. The values of r , s , and C were first computed for each panel by using an average K for the columns. Then for the panels between the 23rd and 24th floors $.222 V_1 = .264 V_2$ or $V_2 = .84 V_1$ and $2 V_1 + V_2 = 37.54^k$ from which $V_1 = 13.22^k$, $V_2 = 11.1^k$. In the first panel the primary moments will be

$$m' = -\frac{(3 + 1.1)(13.22)(11.25)}{11.34 \cdot 2} = -26.8 \text{ ft. kips}$$

and
$$m'' = -\frac{(3 + 4.24)(13.22)(11.25)}{11.34 \cdot 2} = -47.5 \text{ ft. kips.}$$

The vertical corrections will be:

$$\Delta M' = +\frac{(4.24)(27.4)}{11.34} - \frac{(1.10)(50.3)}{11.34} = +10.2 - 4.9 = +5.3$$

at the top and $\Delta M'' = -10.2 + 4.9 = -5.3$ at the bottom. The horizontal correction is made after the vertical correction are all completed, and for the bottom of the panel will be:

$$\Delta M'' = (-0.369 + 0.264)\left(\frac{39.6}{2}\right) + (0.165 - 0.264)\left(\frac{22.9}{2}\right) = -3.2$$

and
$$(+0.369 + 0.264)\left(\frac{39.6}{2}\right) + (-0.165 - 0.264)\left(\frac{22.9}{2}\right) = +7.6.$$

The moments in the girders were computed from the column moments by distributing the sum of the column moments at each joint to the girders in proportion to their rigidity. The figures shown in parentheses (Fig. 13) are the corresponding values computed by the moment distribution method.

The above panel method will usually give satisfactory results when the rigidity of the columns in each story is fairly constant and the arrangement of the members is regular. For more accurate results the more laborious exact methods can be utilized in which only a portion of the structure is used, or the moment distribution method as already applied to the Vierendeel truss will be found useful.

Summary.

In this paper the more recent methods of successive approximations are presented, with particular emphasis upon two types; one, such as the moment distribution method, in which each member is considered as a primary unit of the structure and, a second type, called the panel method, in which the various panels are taken as the primary structural units. The first type is very useful for analyzing those structures in which the joints undergo rotation, but when the joints have considerable displacement, as in the Vierendeel systems, then this type of method gives a rather involved and indirect solution for the majority of problems. The panel method, however, is particularly applicable to many structures of the Vierendeel type and, as the numerical examples show, it provides a direct solution with very little numerical work. Several early methods of analysis are briefly mentioned to show more clearly their place in the development of the above solutions.

In the panel method, simple formulae are given for the calculation of the moments in Vierendeel trusses with chords of equal rigidity and these formulae are then applied to numerical examples of viaducts, bridges and buildings. A discussion accompanied by some numerical data is also given of the effect of the change in length of the chords, which was neglected in the first set of equations, and of variations in the truss proportions upon the bending moments.

When these two methods of successive approximations are thoroughly understood, they provide an easy and accurate solution for many types of rigid frame structures.

Résumé.

Le but du présent mémoire est d'exposer les méthodes d'approximations successives les plus récemment proposées, en insistant tout particulièrement sur deux principes généraux: le premier, constituant la méthode de distribution des moments, dans laquelle chaque élément est considéré comme élément primaire de l'ouvrage; le second, constituant la méthode des panneaux, dans lesquels ce sont les différents panneaux qui sont considérés comme éléments primaires de l'ouvrage.

La première méthode est très utile pour l'étude analytique des ouvrages dans lesquels les assemblages sont exposés à des rotations; toutefois, lorsque comme dans les systèmes Vierendeel les assemblages ont à supporter des déformations très importantes, cette méthode ne donne dans la majorité des cas que des solutions plutôt compliquées et indirectes. Par contre la méthode des panneaux est applicable tout particulièrement à de nombreux ouvrages du type Vierendeel; comme le montrent les exemples numériques, elle permet d'obtenir une solution directe au prix de calculs numériques très peu compliqués.

L'auteur mentionne également brièvement plusieurs méthodes antérieures d'analyse, afin de mettre plus nettement en évidence le rôle qu'elles ont joué dans le développement des deux méthodes principales ci-dessus.

Dans la méthode des panneaux, l'auteur donne des formules simples pour le calcul des moments dans les systèmes Vierendeel avec membrures d'égale rigidité; puis il en montre l'application à des exemples numériques de calcul de viaducs, de ponts et de charpentes. Une étude critique, accompagnée de données numériques, porte également sur l'influence des modifications de longueur des membrures, point qui a été laissé de côté dans les équations précédentes, ainsi que sur l'influence des variations dans les proportions des systèmes sur les moments fléchissants.

La mise en oeuvre parfaitement judicieuse de ces deux méthodes permet de déterminer facilement et avec précision la solution convenant à de nombreux types d'ouvrages rigides.

Zusammenfassung.

Die vorliegende Arbeit behandelt die neuesten Methoden der sukzessiven Annäherung unter besonderer Hervorhebung von zwei Typen; erstens der Momentenverteilungsmethode, nach der jedes Konstruktionsglied als primäre Einheit betrachtet wird, und zweitens die Feldmethode, nach der die verschiedenen Felder als primäre Konstruktionseinheiten angesprochen werden. Die erste Methode eignet sich besonders zur Berechnung von Konstruktionen, bei denen die Knotenpunkte Drehungen erleiden; verschieben sich aber die Knotenpunkte um ein beträchtliches Maß, wie beispielsweise bei Vierendeel-systemen, dann ergibt diese Methode eine ziemlich verwickelte und indirekte Lösung für die meisten Probleme. Die Feldmethode hingegen ist hauptsächlich für viele Konstruktionen des Vierendeel-Typus anwendbar, und, wie die

numerischen Beispiele zeigen, liefert eine direkte Lösung mit geringem Rechenaufwand. Einige der früheren Berechnungsmethoden sind kurz erwähnt, um ihre Bedeutung in der Entwicklung der obigen Lösungen deutlicher zu zeigen.

In der Feldmethode werden einfache Formeln entwickelt zur Berechnung der Momente in Vierendeelträgern mit Gurtungen von gleicher Steifigkeit, die bei der Berechnung von numerischen Beispielen für Viadukte, Brücken und Hochbauten angewendet werden.

Im weiteren wurde, erweitert durch numerische Beispiele, die Wirkung der Längenänderung der Gurtungen betrachtet, was in den früheren Gleichungen vernachlässigt wurde, sowie der Einfluß der Änderungen der Trägerabmessungen auf die Biegemomente.

Diese beiden Methoden der sukzessiven Annäherung stellen, wenn sie richtig aufgefaßt werden, eine leichte und genaue Lösung für viele Typen von Steifrahmenkonstruktionen dar.

